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Differential Equations In Science And Engineering | 22/23 IIa
Examination | 26.06.2023

Allowed Assistance:

- Pen, which is not writing red. No pencil.
- one double-sided, handwritten A4 paper sheet, including your name and student number.
- Additional assistance, calculators, mobile phones, are not allowed.

Hints:

- You have **120 minutes** for your examination. All answers have to be explained in detail.
- To pass the examination you need **55%** of the available points.
- Please start every exercise on the sheet, where the task is written on. If you are using additional sheets, note which exercise you refer to at the top of that sheet and write your name and student number on it.

Student number: ___ ___ ___ ___ ___ ___ ___

Name, First Name: ____________________

Signature: ____________________

Exercise	1	2	3	4	$\sum$
Points	5.0	5.0	5.0	5.0	20
Your Points					

Exercise 1.

Let $k(t)$ be the number of kangaroos in Australia at time t (the prey) and $p(t)$ the number of predators at time t. This prey and predator situation can be described by the system

$$\begin{aligned}\frac{dk}{dt} &= \alpha k - \beta k^2 - \gamma k p,\\ \frac{dp}{dt} &= -\sigma p + \lambda k p,\end{aligned}$$

where $\alpha, \beta, \gamma, \sigma, \lambda \in \mathbb{R}_+$ are non-negative constants.

a) What are the physical interpretations of the constants $\alpha, \beta, \gamma, \sigma, \lambda$?

b) From now on we assume that $\beta = 0$. What are the two steady states of the ODE system?

c) Compute the stability properties of the steady states of the system. Classify the steady states.

d) Around the non-trivial steady-state, the solution behaves in cycles. Let us assume that the period of such a cycle is T. Show that the average population $\overline{k}, \overline{p}$ during one cycle, defined as

$$\overline{k} = \frac{1}{T}\int_0^T k(t)\,dt, \quad \overline{p} = \frac{1}{T}\int_0^T p(t)\,dt$$

is given by the non-trivial steady-state itself.

What does that mean for the application?

0.5+1+1.5+2 points

Name: Student number:

Exercise 2.

We consider the following system (1) of chemical reactions for the four species A, B, C, D:

$$
\begin{aligned}
A + B &\xrightarrow{k_1} B + C \\
B + C &\xrightarrow{k_2} 2B \\
C &\xrightarrow{k_3} D
\end{aligned}
\tag{1}
$$

a) Derive the corresponding system of ODEs that describes the dynamics of the species' concentrations denoted by A, B, C, D.

b) Draw the reaction network.

c) Show that $A + B + C + D$ is a conserved quantity.

2.5+1+1.5 points

Name: Student number:

Exercise 3.

We consider the scalar wave equation

$$\frac{\partial^2}{\partial t^2}u - c\frac{\partial^2}{\partial x^2}u = 0, \quad c \in \mathbb{R}, \tag{2}$$

with constant wave velocity c.

We want to perform a linear stability analysis of equation (2) using the wave ansatz

$$u(t,x) = c \cdot e^{i(kx-\omega t)}, \tag{3}$$

for wave number $k \in \mathbb{R}$, wave frequencies $\omega \in \mathbb{C}$ and amplitude $c \in \mathbb{R}$.

a) What wave frequencies ω in (3) lead to a stable wave in time?

b) Insert the wave ansatz (3) into the wave equation (2) to derive a stability condition for the wave equation. Show that the stability condition is equivalent to $c > 0$.

c) What is the physical interpretation of this stability condition and does it make sense?

1+3+1 points

Exercise 4.

The shallow water equations for water height $h(t,x)$ and vertical velocity $u(t,x)$ are

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \cos(\alpha) g \frac{h^2}{2} \end{pmatrix} = -\frac{1}{\lambda} \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{4}$$

where $h(t,x)$ and $u(t,x)$ are the unknowns and g, λ are parameters.

a) What physical interpretations do the equations (4) have and what are the main assumptions for their derivation?

b) Show that the system (4) can be written in the following (so-called primitive variable) form:

$$\partial_t \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} u & h \\ \cos(\alpha) g & u \end{pmatrix} \cdot \partial_x \begin{pmatrix} h \\ u \end{pmatrix} = -\frac{1}{\lambda h} \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{5}$$

c) Assume the spatially homogeneous case in which all spatial derivatives vanish, i.e. $\partial_x h = 0$, $\partial_x u = 0$. Compute the solution of the shallow water equations (4).

d) What problems can appear for numerical schemes trying to solve the homogeneous shallow water equations?

1.5+1.5+1.5+0.5 points

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